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Vector Risk Functions

Alejandro Balbás, Raquel Balbás and Pedro Jiménez-Guerra*

Abstract. The paper introduces a new notion of vector-valued risk function, a crucial notion in Actuarial and Financial Mathematics. Both deviations and expectation bounded or coherent risk measures are defined and analyzed. The relationships with both scalar and vector risk functions of previous literature are discussed, and it is pointed out that this new approach seems to appropriately integrate several preceding points of view. The framework of the study is the general setting of Banach lattices and Bochner integrable vector-valued random variables. Sub-gradient linked representation theorems and practical examples are provided.

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Keywords. Vector risk function, representation theorem, dynamic risk measures and other examples.

1. Introduction

The notion of coherent measure of risk was introduced in the seminal paper by Artzner *et al.* (1999), and since then their work has been extended in many directions. Jouini *et al.* (2004) justified the use of vector random variables to represent the final wealth provided by some portfolios, as well as the use of "coherent vector-valued risk measures" to reflect risk levels. Cascos and Molchanov (2007) enlarged the set of financial applications of these new frameworks.

The interest of the approach of Jouini *et al.* (2004) justifies possible extensions of their discussion so as to incorporate much more practical situations. For instance, they deal with a L_∞ space, whereas many scalar coherent risk measures are defined on a larger L_p space (for example, L_1 is the natural

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space to introduce the Conditional Value at Risk). In this line, Hamel and Heyde (2010) show that the approach also makes sense in more general L_p spaces, and Balbás and Jiménez-Guerra (2010) deal with risks and risks measures which are valued in a general Banach Lattice, rather than the classical finite-dimensional space $\mathbb{R}^n$.

Besides, while Artzner *et al.* (1999) understood their risk measures as initial capital requirements that investors and managers should provide in order to overcome negative evolutions of the market, recent literature has shown the interest of drawing on risk measures in order to address other classical topics, such as pricing and hedging issues (Nakano, 2004) or portfolio choice problems (Ogryczak and Ruszczynski, 1999, Benati, 2003, Konno *et al.*, 2005, Rockafellar *et al.*, 2006*a*, etc.). This fact has led to further studies concerning risk analysis, and the use of convex measures (Föllmer and Schied, 2002), consistent measures (Goovaerts *et al.* ,2004) or deviations and expectation bounded risk measures (Rockafellar *et al.*, 2006*b*), amongst many other kinds of risk functions. It seems that extending the Jouini *et al.* (2004) analysis makes it easier to deal with the issues above under weaker restrictions.

This paper aims to present a general framework of vector risk functions. We introduce a "generalized vector risk function" as a map $\rho : L_p(\mu, E) \rightarrow F$, with μ being a probability and E and F being general Banach lattices (Meyer-Nieberg, 1991). According to the properties of ρ, we define "coherent measures", "deviations", and "expectation bounded measures". The main difference with respect to previous literature dealing with vector risk measures is that we will not deal with "set-valued functions". On the contrary $\rho(y)$ is not a complex subset of F but a single element, for every $y \in L_p(\mu, E)$. This new approach significantly simplifies previous ones and retrieves suitable and natural properties; For instance, the simultaneous consideration of scalar deviations or coherent expectation bounded risk measures generates vector deviations or vector coherent expectation bounded risk functions. Simultaneously, we deal with a general framework, since we do not impose finite dimensions and p is arbitrary within the interval $[1, \infty]$.

The outline of the paper is as follows. Section 2 introduces the general setting and those previous concepts and properties that we will need throughout the article. Section 3 introduces the generalized vector risk functions, their properties and some important relationships. Section 4 presents Representation Theorems. We have followed the idea of Rockafellar *et al.* (2006*b*), in the sense that we represent the measure ρ "as an envelope of its sub-gradients", which, as long as E satisfies the Radon-Nikodym property (Diestel and Uhl, 1977), are elements of $L_q(\mu, E^*)$; q being the conjugate of p and E^* denoting the dual space of E. Section 5 presents some practical examples and financial applications of vector risk functions, with special focus on dynamic risk measures. Both, scalar and vector dynamic risk measures are particular cases of the vector risk functions that we will introduce in the third section. Section 6 concludes the article.

2. Preliminaries and notations

Throughout the paper, E, F, E_+ and F_+ will denote two Banach lattices and their non-negative cones respectively.[1] Their dual Banach lattices and cones will be represented by E^*, F^*, E_+^* and F_+^*, and $\langle e^*, e\rangle$ will be "the usual product" of $e^* \in E^*$ and $e \in E$. If $e_1, e_2 \in E$ and $e_1 - e_2 \in E_+$ then we will write $e_1 \geq e_2$. Similar ideas apply if F plays the role of E.

$\mathcal{L}(E, F)$ ($\mathcal{L}_+(E, F)$) will be the set of linear maps $\Lambda : E \to F$ that are continuous (non-negative, *i.e.*, $\Lambda(e) \geq 0$ whenever $e \geq 0$). Every $\Lambda \in \mathcal{L}_+(E, F)$ is continuous (Meyer-Nieberg, 1991).

$(\Omega, \mathcal{F}, \mu)$ will be a probability space composed of the set Ω, the σ-algebra $\mathcal{F}$ and the probability measure μ. $p \in [1, \infty]$ and $q \in [1, \infty]$ will be conjugate values, *i.e.*, $1/p + 1/q = 1$. If $p < \infty$ then $L_p(\mu, E)$ will represent the Banach space of those Bochner integrable (Diestel and Uhl, 1977) functions $y : \Omega \to E$ such that $\int_\Omega \|y(\omega)\|^p d\mu(\omega) < \infty$, endowed with the usual norm

$$\|y\|_p = \left(\int_\Omega \|y(\omega)\|^p d\mu(\omega)\right)^{\frac{1}{p}}.$$

Similarly, $L_\infty(\mu, E)$ will be the Banach space of E-valued essentially bounded and integrable functions, endowed with the norm

$$\|y\|_\infty = ess - \sup \ \{\|y(\omega)\|; \omega \in \Omega\}$$

$ess - \sup$ denoting the essential supremum. It is well known that $L_{p_1}(\mu, E) \subset L_{p_2}(\mu, E)$ whenever $p_1 \geq p_2$ and the natural inclusion is continuous. If $p < \infty$ and E^* satisfies the Radon-Nikodym property then, $L_q(\mu, E^*)$ is the dual space of $L_p(\mu, E)$. Henceforth we will assume that E and E^* satisfy the Radon-Nikodym property (Diestel and Uhl, 1977).

If $\mathcal{G}$ denotes a sub-σ-algebra of $\mathcal{F}$ and $\mu_\mathcal{G}$ is the restriction of μ to $\mathcal{G}$, then $L_p(\mu_\mathcal{G}, E)$ is a closed subspace of $L_p(\mu, E)$. In such a case $\mathbb{E}(y\,|\mathcal{G}) \in L_p(\mu_\mathcal{G}, E)$ for every $y \in L_p(\mu, E)$, $\mathbb{E}(y\,|\mathcal{G})$ denoting the conditional expectation of y with respect to $\mathcal{G}$.

An interesting particular example is $\mathcal{G} = \{\emptyset, \Omega\}$, in which case $L_p(\mu_\mathcal{G}, E)$ and E may be identified. Indeed, if there is no confusion, for every $y_0 \in E$ we will also represent by y_0 the constant element of $L_p(\mu, E)$ given by $y(\omega) = y_0$ *a.s.* Furthermore, $\mathbb{E}(y\,|\mathcal{G})$ will be represented by $\mathbb{E}(y) = \int_\Omega y d\mu \in E$, for every $y \in L_p(\mu, E)$.

3. Generalized vector risk functions

Definition 3.1. Every

$$\rho : L_p(\mu, E) \to F$$

[1] Most of the properties here stated would still hold if E or F were ordered Banach spaces. However we think that imposing E and F to be Banach lattices the exposition is significantly simplified.

will be called Vector Risk Function (VRF). Furthermore, ρ is said to be:

a) $(\Lambda, \mathcal{G})$-Translation invariant, if $\mathcal{G}$ denotes a sub-σ-algebra of $\mathcal{F}$,

$$\Lambda : L_p(\mu_{\mathcal{G}}, E) \longrightarrow F$$

is linear and continuous, and $\rho(y + y_0) = \rho(y) - \Lambda(y_0)$ holds for every $y \in L_p(\mu, E)$ and every $y_0 \in L_p(\mu_{\mathcal{G}}, E)$.[2] If $\mathcal{G} = \{\emptyset, \Omega\}$ (*i.e.*, $L_p(\mu_{\mathcal{G}}, E) \approx E$) and $\Lambda \in L_+(E, F)$, then we will merely say that ρ is Λ-translation invariant.

b) Positively homogeneous, if $\rho(\alpha y) = \alpha \rho(y)$ holds for every real number $\alpha > 0$ and every $y \in L_p(\mu, E)$.

c) Sub-additive, if $\rho(y_1 + y_2) \leq \rho(y_1) + \rho(y_2)$ holds for every $y_1, y_2 \in L_p(\mu, E)$.

d) Decreasing, if $\rho(y_2) \leq \rho(y_1)$ whenever $y_1, y_2 \in L_p(\mu, E)$ and $y_2 \geq y_1$ *a.s.*

e) $(\Lambda, \mathcal{G})$-Mean dominating, if $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$, $\Lambda : L_p(\mu_{\mathcal{G}}, E) \longrightarrow F$ is linear and continuous, and $\rho(y) \geq -\Lambda(\mathbb{E}(y|\mathcal{G}))$ holds for every $y \in L_p(\mu, E)$. □

Definition 3.2. The VRF ρ is said to be:

a) A $(\Lambda, \mathcal{G})$-expectation bounded risk measure, if $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$, $\Lambda : L_p(\mu_{\mathcal{G}}, E) \longrightarrow F$ is linear and continuous, and ρ is $(\Lambda, \mathcal{G})$-translation invariant, positively homogeneous, sub-additive, and $(\Lambda, \mathcal{G})$-mean dominating.

b) A $(\Lambda, \mathcal{G})$-coherent risk measure, if $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$, $\Lambda : L_p(\mu_{\mathcal{G}}, E) \longrightarrow F$ is linear and continuous, and ρ is $(\Lambda, \mathcal{G})$-translation invariant, positively homogeneous, sub-additive, and decreasing.

c) A $\mathcal{G}$-deviation if $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$ and ρ is $(\Lambda, \mathcal{G})$-expectation bounded with $\Lambda = 0$. □

Remark 3.3. Definition 3.2. above extends the notion of scalar coherent risk measure (Artzner *et al.*, 1999), scalar expectation bounded risk measure (Rockafellar *et al.*, 2006*b*), and scalar deviation measure (Rockafellar *et al.*, 2006*b*). Scalar measures arise when $E = F = \mathbb{R}$, $\mathcal{G} = \{\emptyset, \Omega\}$, and Λ is the identity map. □

Proposition 3.4. *Let ρ be a VRF.*

a) *If ρ is positively homogeneous then $\rho(0) = 0$.*

b) *If ρ is $(\Lambda, \mathcal{G})$-expectation bounded or $(\Lambda, \mathcal{G})$-coherent for some $(\Lambda, \mathcal{G})$ then $\rho(y_0) = -\Lambda(y_0)$ for every $y_0 \in L_p(\mu_{\mathcal{G}}, E)$.*[3]

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[2] Following Artzner *et al.* (1999), if Λ is onto we can consider that, given $y \in L_p(\mu, E)$, any $y_0 \in L_p(\mu_{\mathcal{G}}, E)$ such that $\Lambda(y_0) = \rho(y)$ may be understood as a final wealth or pay-off that must be guaranteed by the initial capital requirements. Indeed, one has that

$$\rho(y + y_0) = \rho(y) - \Lambda(y_0) = 0,$$

so the global risk vanishes with the additional wealth y_0.

[3] In particular, $\rho(y_0) = 0$ for every $\mathcal{G}$-deviation ρ and every $y_0 \in L_p(\mu_{\mathcal{G}}, E)$. Notice that it is sufficient to impose ρ to be $(\Lambda, \mathcal{G})$-translation invariant and positively homogeneous.

Proof. To prove a) notice that $\rho(0) = \rho(\alpha 0) = \alpha\rho(0)$, so $\rho(0) \neq 0$ would lead to $\alpha = 1$ for every positive α.

To prove b) notice that $\rho(y_0) = \rho(0 + y_0) = \rho(0) - \Lambda(y_0) = -\Lambda(y_0)$. □

The following result establishes the existence of a one to one mapping between deviations and expectation bounded risk measures.

Proposition 3.5. *Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$ and $\Lambda : L_p(\mu_{\mathcal{G}}, E) \longrightarrow F$ linear and continuous. The relationship*

$$\rho \to D = \rho + \Lambda \circ \mathbb{E}(- |\mathcal{G})$$

establishes a one to one correspondence between the set of $(\Lambda, \mathcal{G})$-expectation bounded risk measures and the set of $\mathcal{G}$-deviations.

Proof. If ρ is a $(\Lambda, \mathcal{G})$-expectation bounded risk measure then set $D = \rho + \Lambda \circ \mathbb{E}(- |\mathcal{G})$ and D is trivially $(0, \mathcal{G})$-translation invariant, positively homogeneous and sub-additive. To show that D is $(0, \mathcal{G})$-mean dominating, take $y \in L_p(\mu, E)$. Then,

$$D(y) = \rho(y) + \Lambda \circ \mathbb{E}(y |\mathcal{G}) \geq 0$$

because ρ is $(\Lambda, \mathcal{G})$-mean dominating and $\rho(y) \geq -\Lambda \circ \mathbb{E}(y |\mathcal{G})$.

Conversely, suppose that D is a deviation and set $\rho = D - \Lambda \circ \mathbb{E}(- |\mathcal{G})$. ρ is clearly $(\Lambda, \mathcal{G})$-translation invariant, positively homogeneous and sub-additive. To show that ρ is $(\Lambda, \mathcal{G})$-mean dominating, take $y \in L_p(\mu, E)$, and one has that

$$\rho(y) = D(y) - \Lambda \circ \mathbb{E}(y |\mathcal{G}) \geq -\Lambda \circ \mathbb{E}(y |\mathcal{G})$$

because $D(y) \geq 0$. □

4. Representation theorems

Artzner *et al.* (1999) and Jouini *et al.* (2004) stated Representation Theorems of "their coherent risk measures" (scalar and vector, respectively) by using duality properties and μ-continuous finitely or σ-finitely additive measures on the measurable space $(\Omega, \mathcal{F})$. The extensions of Hamel and Heyde (2010) and Balbás and Jiménez-Guerra (2010) also led to analogous (but more general) Representation Theorems. However, Rockafellar *et al.* (2006*b*) represented "their (real-valued) expectation bounded risk measures" by using $L_2(\mu, \mathbb{R})$, which may be identified with its dual space. Here we draw on the duality $(L_q(\mu, E^*), L_p(\mu, E))$ and follow the ideas of the Rockafellar *et al.* (2006*b*) in order to represent the VRF by "some kind of envelope generated by its sub-gradients".

Lemma 4.1. *Suppose that $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$, $p < \infty$ and $F = \mathbb{R}$. If $D : L_p(\mu, E) \to \mathbb{R}$ is a real valued and continuous $\mathcal{G}$-deviation then there exists $\Delta \subset L_q(\mu, E^*)$ satisfying the following conditions:*

a) Δ is convex and $\sigma(L_q(\mu, E^*), L_p(\mu, E))$-compact.

b) The equality

$$D(y) = Max \left\{ -\int_{\Omega} \langle z^*(\omega), y(\omega) \rangle \, d\mu(\omega) \, ; z^* \in \Delta \right\} \tag{4.1}$$

holds for every $y \in L_p(\mu, E)$.

Proof. Since E^* satisfies the Radon-Nikodym property we have that $L_q(\mu, E^*)$ is the dual space of $L_p(\mu, E)$. Besides, if

$$\Delta_1 = \left\{ z^* \in L_q(\mu, E^*) \, ; D(y) \geq \int_{\Omega} \langle z^*(\omega), y(\omega) \rangle \, d\mu(\omega), \ \forall y \in L_p(\mu, E) \right\}$$

then it may be easily proved that Δ_1 is convex and $\sigma(L_q(\mu, E^*), L_p(\mu, E))$-closed. Furthermore, since $L_q(\mu, E^*)$ is the dual space of $L_p(\mu, E)$ and D is continuous, Theorem 2.4.9 in Zalinescu (2002) implies that Δ_1 is $\sigma(L_q(\mu, E^*), L_p(\mu, E))$-compact, along with the equality

$$D(y) = Sup \left\{ \int_{\Omega} \langle z^*(\omega), y(\omega) \rangle \, d\mu(\omega) \, ; z^* \in \Delta_1 \right\}$$

for every $y \in L_p(\mu, E)$. Hence, the result trivially follows if one takes $\Delta = -\Delta_1$. □

Remark 4.2. It is worth pointing out that every $z^* \in \Delta$ satisfies

$$\mathbb{E}(z^* \, |\mathcal{G}) = 0. \tag{4.2}$$

Indeed, Proposition 3.4.*b*) and the proof of Lemma 4.1. imply that

$$-\int_{\Omega} \langle z^*, y_0 \rangle \, d\mu \leq D(y_0) = 0$$

holds for every $y_0 \in L_p(\mu_{\mathcal{G}}, E)$. Thus, if $-y_0$ replaces y_0 we have

$$\int_{\Omega} \langle z^*, y_0 \rangle \, d\mu = 0.$$ □

Lemma 4.3. *Suppose that $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$, $p < \infty$, $F = \mathbb{R}$ and $\Lambda : L_p(\mu_{\mathcal{G}}, E) \to \mathbb{R}$ is linear and continuous. If $\rho : L_p(\mu, E) \to \mathbb{R}$ is a real valued $(\Lambda, \mathcal{G})$-expectation bounded and continuous risk measure then there exist $\Delta \subset L_q(\mu, E^*)$ and $\Lambda_{\mathcal{G}} \in L_q(\mu_{\mathcal{G}}, E^*)$ satisfying Condition a) above and*

$$\rho(y) = Max \left\{ -\int_{\Omega} \langle z^*(\omega), y(\omega) \rangle \, d\mu(\omega) \, ; z^* \in \Delta \right\} - \int_{\Omega} \langle \Lambda_{\mathcal{G}}(\omega), y(\omega) \rangle \, d\mu(\omega) \tag{4.3}$$

holds for every $y \in L_p(\mu, E)$. Furthermore, $z^*(\omega) + \Lambda_{\mathcal{G}} \geq 0$ a.s. for every $z^* \in \Delta$ if and only if ρ is $(\Lambda, \mathcal{G})$-coherent.

Proof. Proposition 3.5. and Expression ((4.1)) trivially imply the existence of $\Delta \subset L_q(\mu, E^*)$ satisfying Condition a) above and such that

$$\rho(y) = Sup \left\{ -\int_{\Omega} \langle z^*(\omega), y(\omega) \rangle \, d\mu(\omega) \, ; z^* \in \Delta \right\} - \Lambda(\mathbb{E}(y \, |\mathcal{G}))$$

for every $y \in L_p(\mu, E)$. Thus, if $\Lambda_{\mathcal{G}} \in L_q(\mu, E^*)$ is such that

$$\int_{\Omega} \langle \Lambda_{\mathcal{G}}(\omega), y(\omega) \rangle \, d\mu(\omega) = \Lambda(\mathbb{E}(y|\mathcal{G}))$$

for every $y \in L_p(\mu, E)$ then ((4.3)) becomes obvious.

Suppose that $z^*(\omega) + \Lambda_{\mathcal{G}} \geq 0$ $a.s.$ for every $z^* \in \Delta$. Then, take $y_1, y_2 \in L_p(\mu, E)$ with $y_2 \geq y_1$ $a.s.$ and

$$-\int_{\Omega} \langle z^*(\omega) + \Lambda_{\mathcal{G}}(\omega), y_1(\omega) \rangle \, d\mu(\omega) \geq -\int_{\Omega} \langle z^*(\omega) + \Lambda_{\mathcal{G}}(\omega), y_2(\omega) \rangle \, d\mu(\omega)$$

trivially holds. Thus, ρ is $(\Lambda, \mathcal{G})$-coherent due to ((4.3)).

Conversely, suppose that $\mu\left(\left\{\omega \in \Omega; z^*(\omega) + \Lambda_{\mathcal{G}}(\omega) \notin E_+^*\right\}\right) > 0$ for some $z^* \in \Delta$. Then, there exists $y \in L_p(\mu, E)$, $y \geq 0$ $a.s.$, such that

$$\int_{\Omega} \langle z^*(\omega) + \Lambda_{\mathcal{G}}(\omega), y(\omega) \rangle \, d\mu(\omega) < 0.$$

Hence, ((4.3)) leads to $\rho(y) > 0$, $i.e.$, according to Proposition 3.4.a), $\rho(y) > \rho(0)$, and ρ is neither decreasing nor $(\Lambda, \mathcal{G})$ $-$coherent. □

Remark 4.4. Notice that $\{\Lambda_{\mathcal{G}}\} + \Delta$ may play the role of Δ in the latter lemma, in which case, we would obtain the existence of Δ, convex and $\sigma(L_q(\mu, E^*), L_p(\mu, E))$-compact, satisfying

$$\rho(y) = Max\left\{-\int_{\Omega} \langle z^*(\omega), y(\omega) \rangle \, d\mu(\omega); z^* \in \Delta\right\} \tag{4.4}$$

for every $y \in L_p(\mu, E)$, and ρ is $(\Lambda, \mathcal{G})$-coherent if and only if $z^*(\omega) \geq 0$ $a.s.$ for every $z^* \in \Delta$. Notice also that ((4.2)) leads to the equality

$$\mathbb{E}(z^*|\mathcal{G}) = \Lambda_{\mathcal{G}}$$

for every $z^* \in \Delta$. □

Theorem 4.5 (Representation Theorem for Deviations). *Suppose $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$ and $p < \infty$. If $D : L_p(\mu, E) \to F$ is a continuous $\mathcal{G}$-deviation then for every $f^* \in F_+^*$ there exists $\Delta_{f^*} \subset L_q(\mu, E^*)$ satisfying the following conditions:*

a) Δ_{f^*} is convex and $\sigma(L_q(\mu, E^*), L_p(\mu, E))$-compact.

b) The equality

$$f^* \circ D(y) = Max\left\{-\int_{\Omega} \langle z^*(\omega), y(\omega) \rangle \, d\mu(\omega); z^* \in \Delta_{f^*}\right\}$$

holds for every $y \in L_p(\mu, E)$.

c) $\mathbb{E}(z^*|\mathcal{G}) = 0$ holds for every $z^* \in \Delta_{f^*}$.

Proof. It is a trivial consequence of Lemma 4.1. and Remark 4.2., if one takes into consideration that $f^* \circ D$ satisfies the properties of an $\mathbb{R}-$valued $\mathcal{G}-$deviation. □

Lemma 4.6. *If $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$, $\Lambda : L_p(\mu_{\mathcal{G}}, E) \longrightarrow F$ is linear and continuous, and $\rho : L_p(\mu, E) \to F$ is a $(\Lambda, \mathcal{G})$-expectation bounded VRF, then ρ is $(\Lambda, \mathcal{G})$-coherent if and only if $f^* \circ \rho$ is decreasing for every $f^* \in F_+^*$.*

Proof. The result is clear if one bears in mind that F is a Banach lattice and, thus, for $f_1, f_2 \in F$ we have that $f_1 \leq f_2$ if and only if $\langle f^*, f_1 \rangle \leq \langle f^*, f_2 \rangle$ for every $f^* \in F_+^*$. $\square$

Theorem 4.7 (Representation Theorem for Expectation Bounded VRF). *Suppose that $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$, $\Lambda : L_p(\mu_{\mathcal{G}}, E) \longrightarrow F$ is linear and continuous and $p < \infty$. If $\rho : L_p(\mu, E) \to F$ is a $(\Lambda, \mathcal{G})$-expectation bounded and continuous VRF then for every $f^* \in F_+^*$ there exist $\Delta_{f^*} \subset L_q(\mu, E^*)$ and $\Lambda_{\mathcal{G}} \in L_q(\mu_{\mathcal{G}}, E^*)$ satisfying the following conditions:*

a) Δ_{f^*} is convex and $\sigma - (L_q(\mu, E^*), L_p(\mu, E))$-compact.

b) The equality

$$f^* \circ \rho(y) = Max \left\{ -\int_\Omega \langle z^*(\omega), y(\omega) \rangle \, d\mu(\omega) ; z^* \in \Delta_{f^*} \right\}$$

holds for every $y \in L_p(\mu, E)$.

c) $\mathbb{E}(z^* | \mathcal{G}) = \Lambda_{\mathcal{G}}$ for every $z^* \in \Delta_{f^*}$.

Moreover, ρ is $(\Lambda, \mathcal{G})$-coherent if and only if $z^* \geq 0$ a.s. for every $f^* \in F_+^*$ and every $z^* \in \Delta_{f^*}$.

Proof. It is a trivial consequence of Lemma 4.6. and Remark 4.4., if one takes into consideration that $f^* \circ \rho$ satisfies the properties of an $\mathbb{R}-$valued $(f^* \circ \Lambda, \mathcal{G})$-expectation bounded risk measure. $\square$

5. Examples and applications

Example 1. It is worth pointing out that one can obtain vector risk functions by simultaneously considering several scalar risk functions. One might consider that this natural property should obviously hold, but recall that this is false for the alternative approach dealing with set-valued vector risk measures (Jouini *et al.*, 2004, Cascos and Molchanov, 2007, Balbás and Jiménez-Guerra, 2010, or Hamel and Heyde, 2010).

More generally, consider the set of Banach lattices E and $\{F_j\}_{j=1}^n$ where E and E^* satisfy the Radon-Nikodym property, the sub-σ-algebra $\mathcal{G}$ of $\mathcal{F}$, the family of continuous linear maps $\Lambda_j : L_p(\mu_{\mathcal{G}}, E) \to F_j$, and the risk functions $\rho_j : L_p(\mu, E) \to F_j$, $j = 1, 2, \ldots, n$. Take

$$\rho : L_p(\mu, E) \to \prod_{j=n}^{n} F_j$$

as usual. Then, if each ρ_j is a $\mathcal{G}$-deviation so is ρ, and if each ρ_j is $(\Lambda_j, \mathcal{G})$-expectation bounded (respectively, $(\Lambda_j, \mathcal{G})$-coherent) then ρ is $(\Lambda, \mathcal{G})$-expectation bounded (respectively, $(\Lambda, \mathcal{G})$-coherent), where

$$\Lambda\left(y_0\right) = \left(\Lambda_j\left(y_0\right)\right)_{j=1}^{n} \in \prod_{j=n}^{n} F_j$$

for every $y_0 \in L_p\left(\mu_{\mathcal{G}}, E\right)$.

Example 2. Jouini *et al.* (2004) provided interesting situations making it quite convenient to consider vector-valued random variables to represent final wealths (or pay-offs) and risk levels (see also Cascos and Molchanov, 2007). For instance, if a portfolio were diversified amongst several currencies and transaction costs made it inefficient to compensate possible losses in a given currency with those profits generated in the remaining ones, then we could take $E = \mathbb{R}^n$ and $F = \mathbb{R}^m$ with $m \leq n$. Then, $y \in L_p\left(\mu, E\right)$ would indicate the pay-off of each sub-portfolio associated with the corresponding currency, and $\rho\left(y\right)$ could indicate the capital requirement to overcome the risk, currency by currency. Of course, m may be less than n because some currencies may be liquid enough so as to accept compensations, and $\mathcal{G} = \{\emptyset, \Omega\}$.

The arguments above also apply in different situations non-necessarily related to currencies. Mainly, one needs a global portfolio that may be divided into sub-portfolios in such a way that it is expensive to compensate among them all due to imperfections.

Example 3.(Dynamic risk measures) Dynamic risk measures are very important in Actuarial and Financial Mathematics because prices, risks and strategies evolve in a stochastic framework. There are several approaches dealing with the notion of dynamic measure of risk, though the most usual one is that in Frittelli and Rosazza Gianin (2004), Cheridito *et al.* (2005) or Roorda and Schumacher (2011), among others. Thus, consider the closed interval $[0, T]$ representing a time period and a set $\mathcal{T} \subset [0, T]$ representing the trading dates and such that $\{0, T\} \subset \mathcal{T}$. The arrival of information is given by the filtration $\left(\mathcal{G}_t\right)_{t \in \mathcal{T}}$ such that $\mathcal{G}_0 = \{\emptyset, \Omega\}$ and $\mathcal{G}_T = \mathcal{F}$. For every adapted stochastic process $\left(S_t\right)_{t \in \mathcal{T}}$ such that $S_t \in L_2\left(\mu_{\mathcal{G}_t}, \mathbb{R}\right)$, usually representing an agent wealth (or loss) at every date $t \in \mathcal{T}$, one can fix $s < t \in \mathcal{T}$ and then, define a risk function $\rho_{(s,t)} : L_2\left(\mu_{\mathcal{G}_t}, \mathbb{R}\right) \to L_2\left(\mu_{\mathcal{G}s}, \mathbb{R}\right)$. Actually the system $\left(\rho_{(s,t)}\right)_{s<t \in \mathcal{T}}$ is a dynamic risk measure, and the analysis fits in our much more general approach if one takes $E = \mathbb{R}$, $F = L_2\left(\mu_{\mathcal{G}s}, \mathbb{R}\right)$, $\mathcal{G} = \mathcal{G}_s$, and $\Lambda : L_2\left(\mu_{\mathcal{G}s}, \mathbb{R}\right) \to L_2\left(\mu_{\mathcal{G}s}, \mathbb{R}\right)$ given by the identity map. Thus, our general definitions of Section 3 and the Representation Theorems of Section 4 apply for dynamic measures of risk. Notice finally that the role of $E = \mathbb{R}$ may be plaid by other vector space, and consequently the approach of this paper applies for both scalar and vector dynamic risk measures. As far as we know, the notion of vector dynamic risk measure had not been introduced yet in the literature.

Example 4.(Optimizing vector risk functions) The optimization of scalar measures of risk is a very important problem in Actuarial and Financial Mathematics because many practical decisions must be optimal. For instance, Benati (2003) and Konno *et al.* (2005) study portfolio choice problems, Schweizer (1995) and Nakano (2004) deal with pricing issues, Balbás *et al.* (2009) analyze optimal reinsurance problems, and Rockafellar *et al.* (2006*a*) and Balbás *et al.* (2010*b*) deal with market equilibrium topics.

In general, the minimization of scalar risk measures is complex in practice because risk measures are not differentiable functions, and the usual Lagrangian linked methods do not apply. Thus, most of the papers above must develop special methods that solve their concrete problem. Nevertheless, there are recent papers whose focus is on the minimization of risk, and they provide general optimization techniques (Balbás *et al.*, 2010*a*, among others). The representation theorems play a crucial role in the development of these general methodologies, since they allow us to find equivalent differentiable optimization problems.

Vector optimization problems are also usual in practical applications. If they are convex, which always holds if one minimizes vector risk measures, then, the most important ways to solve them are the scalarization method and the balance space approach (Galperin, 1997). In both cases, the Representation Theorems of Section 4 permit us to extend the findings of Balbás *et al.* (2010*a*), in such a way that the minimization of vector risks become a differentiable problem, and then, standard Lagrangian linked and saddle point linked necessary and sufficient optimality conditions may be given. We will not present a detailed analysis because this is a straightforward extension of Balbás *et al.* (2010*a*), if one bears in mind Theorems 4.5. and 4.7.

6. Conclusions

The paper has introduced a new notion of vector risk function and concepts such as vector deviation, vector expectation bounded risk measure or vector coherent risk measure. Relationships amongst them have been analyzed. In this sense, the generalized vector risk functions may be used to provide initial capital requirements as well as to deal with most of the classical topics (pricing, hedging, portfolio choice, etc.). The link with dynamic risk functions or vector risk functions studied in previous literature has been discussed, and it has been pointed out that this new approach simplifies many theoretical and practical problems, since we do not deal with set-valued risks. On the contrary, the risk of every (vector) pay-off is a single vector. Practical examples have been illustrated and sub-gradient linked representation theorems have been given.

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